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OPTIMAL DESIGN OF RIGID–PLASTIC CYLINDRICAL SHELLS IN THE POST-YIELD RANGE†

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Abstract—An optimal design technique is developed for rigid–plastic cylindrical shells subjected to a distributed transverse pressure and a specified axial load. Moderately large deflections are taken into account and a deformation-type theory of plasticity is employed. The optimal design procedure results in a unified approach to optimization in the post-yield range. Necessary conditions for optimality are established by the aid of variational methods of the optimal control theory. Two examples are presented: (1) the optimal layout of rigid circular supports (stiffeners) is found which minimizes the mean deflection; (2) the optimal thickness distribution of a sandwich shell is established which corresponds to the minimum material consumption requiring the deflection of the design coinciding with that of the constant thickness shell.

1. INTRODUCTION

Rigid–plastic structures designed under optimality requirements, e.g. the minimum weight at the prescribed collapse loads appear to be sensitive to geometrical changes taking place in the post-yield point range. Configurations have to be taken into account in optimal plastic design as the load supported changes with plastic flow when the structures originally bent must support bending-membrane force interactions. Two features of such designs are disclosed. As the yield point load has no meaning at large displacements, different optimality requirements should be formulated. Moreover, forces and displacements are coupled and strain–displacement relations are nonlinear. As the requirement imposed can be such that the deformed shape remains the same as that at the post-yield range load for uniform structure when configuration changes are taken into account.

For beams, axisymmetric plates and cylindrical shells, optimal thickness variations have been established as well as the optimal locations of additional supports have been obtained in recent years. However, the majority of studies have been restricted to infinitesimal displacements[1–4].

Different approaches to the optimal design of plastic structures operating in the range of finite deflections are suggested in Refs [5–8]. A method for minimum volume design of geometrically non-linear cylindrical shells was developed in Ref. [9] stipulating the deflections of the optimal design associated with the deflections of the uniform shell. This technique was applied in Ref. [10] for the shell subjected to the transverse pressure and axial dead load.

The parametrical optimization procedure developed originally for plastic beams[7] was adopted for cylindrical shells and further refined upon in Refs [11–13]. Contributions [11, 12] are devoted to the particular problems of the determination of optimal locations for additional supports.

This paper deals with the optimization of rigid–plastic cylindrical shells loaded beyond the incipient flow load and preserving the stability of the load–displacement relation. Necessary conditions for optimality are derived employing variational methods of the

† This paper is one of the last publications of Professor Antoni Sawczuk, whose sudden death on 27 May 1984, in his fifty-seventh year is a tragic loss to the world of science.

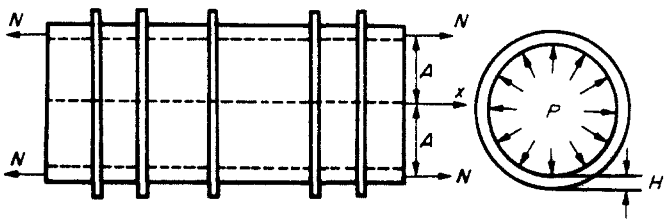

Fig. 1. Shell geometry.

optimal control theory. Mathematically, the question studied becomes that of a control problem which involves the second-order constraints imposed on the state variables, e.g. bending moments, axial forces, displacements. The procedure employed results in a unified approach to the optimization in the post-yield point range.

The principles adopted and the results arrived at are used to establish the optimal layout of rigid, circular stiffeners (supports) for cylindrical closed shells. The optimal layout is found under the requirement that the average deflection attains the minimum value. Finally, the optimal thickness variation is found for a shell under the requirement of the minimum material consumption when deflections of the optimal design are bounded by those relating to the constant thickness shell.

2. FORMULATION OF THE PROBLEM

Consider the moderately large deflections of rigid–plastic circular cylindrical shells of length l and radii A. The shells are subjected to the axial dead load N and to the internal pressure loading of intensity $P(x,p)$ (Fig. 1). The tubes under consideration have a constant or variable wall thickness $H(x,h)$. Here the functions $H = H(x,h)$ and $P = P(x,p)$ are regarded as given differentiable functions whereas h and p may stand for previously unknown constant parameters. The coordinate system with its x-axis coinciding with the undeformed generator of the shell has its origin at the left-hand end of the tube. Assuming that at the positions $x = s_1, \ldots, s_n$, the additional rigid supports may be applied which prevent transverse displacements at these cross-sections of the shell.

The optimal design problems will be considered for which the cost function that is to be minimized is presented as

$$I = G(p,h,s_1,\ldots,s_n)+\int_0^l F(P,H,W,U)\,\mathrm{d}x. \tag{1}$$

In the latter formulae the quantities F and G are regarded as given differentiable functions, whereas U and W stand for the axial and transverse displacement, respectively. Moderately large deflections, e.g. displacements of the order of the shell wall thickness will be admitted.

The minimum of eqn (1) is sought for among the solutions of the governing equations of the geometrically non-linear theory of fully plastic cylindrical shells. The optimal solution is assumed to be constrained by

$$R(P,H,W) \leqslant 0 \tag{2}$$

which is imposed on the stress–strain state of the shell at each point and is given by

$$g_j(p,h,s_1,\ldots,s_n,W(x_j)) \leqslant 0, \qquad j = 1,\ldots,m \tag{3}$$

which are valid at discrete points only.

Differentiability of functions R and g_j ($j = 1,\ldots,m$) with respect to their arguments is expected, as above. Numbers m as well as n are assumed to be specified. These will not be subjected to any variation in the further analysis.

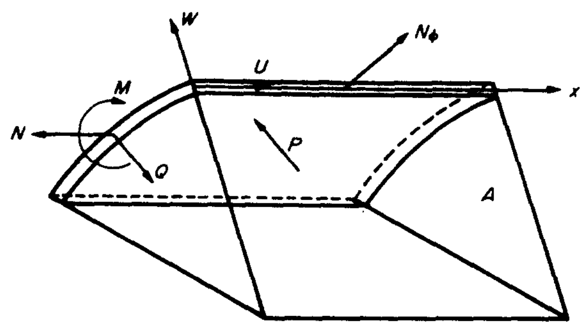

Fig. 2. Shell element (sign convention).

Particular cases of the problem posed above are discussed in Refs [1, 13]. If $G = 0$, $F = W$, $R = 0$, then the problem consists in the minimization of the mean deflection. This should be done by means of determination of a suitable layout of stiffeners, for instance. Another class of problems is associated with $G = 0$, $F = H$, $R = W - W_*(x)$, where $W_*(x)$ is a specified function. Now one has a minimum volume problem with constrained deflections. In this paper $W_*(x)$ will be interpreted as the deflection of the corresponding uniform shell. This concept was used also in Refs [9, 10]. However, function $W_*(x)$ should be an arbitrary non-negative function, provided that the problem is physically meaningful and thus the existence of the desired solution is guaranteed.

3. GOVERNING EQUATIONS AND ASSUMPTIONS

The equilibrium equation of a shell element in the case of moderately large deflections is [14, 15]

$$\frac{d^2M}{dx^2} - N\frac{d^2W}{dx^2} + \frac{N_\varphi}{A} - P = 0 \tag{4}$$

where M and N_φ stand for the bending moment and the circumferential force, respectively. The sign convention, e.g. the positive directions of stress results and displacements are shown in Fig. 2, where Q stands for the shear force. As usual[16–19], the influence of the transverse shear on yielding is neglected.

In the large deflection range the strain–displacement relations have the form[16, 17]

$$\begin{aligned} \varepsilon_x &= \frac{dU}{dx} + \frac{1}{2}\left(\frac{dW}{dx}\right)^2, & \varepsilon_\varphi &= \frac{W}{A}, \\ \kappa_x &= \frac{d^2W}{dx^2}, & \kappa_\varphi &= 0. \end{aligned} \tag{5}$$

Material of the shell is assumed to obey the Tresca yield condition. The effects of both elastic strains and strain-hardening are neglected. Equations of the exact yield surface in the stress resultant space were first derived in Ref. [20] using the assumption of straight normals. Approximations of the exact Tresca yield surface are given and discussed in Refs [21–23]. The strain mapping method suggested by Onat and Prager was further employed in Refs [24–26] to obtain yield surfaces for cylindrical shells with rib reinforcements.

The exact yield surfaces associated with the piece-wise linear Tresca condition have quite intricate structures. As we are more interested in developing a design procedure for plastic shells at large deflection range rather than in a specific solution, we employ approximate yield surfaces.

Such an approximation of the exact yield surface will be used in this paper, for which the stress state of the shell is associated with the plane $N_\varphi = N_0$, only (N_0 denotes the yield force). This hypothesis about the stress profile was introduced in Refs [17, 18]. It was utilized successfully in the plastic analysis as well as in the optimal design of cylindrical shells accounting for large deflections[10–13, 16–18]. It is assumed, thus, that the yield condition is defined by the relations $N_\varphi = N_0$ and

$$\Phi(M, N, h) \leqslant 0 \tag{6}$$

where Φ is a piece-wise smooth function.

A type of deformation theory of plasticity will be used which states that the strain vector with components, eqn (5), is directed along the outward normal to the yield surface. Thus, according to the associated deformation law

$$\varepsilon_x = \lambda^2 \Phi_2, \qquad \kappa_x = \lambda^2 \Phi_1. \tag{7}$$

Here

$$\Phi_1 = \frac{\partial \Phi}{\partial M}, \qquad \Phi_2 = \frac{\partial \Phi}{\partial N} \tag{8}$$

whereas λ^2 denotes a non-negative scalar multiplier, which vanishes, if $\Phi < 0$. Since the stress profile lies on the plane $N_\varphi = N_0$, one has an additional relation $\varepsilon_\varphi = \lambda_1^2$. The latter may be conceived as an equation for determination of the unknown function λ_1; therefore, it will be omitted in the further analysis.

At non-regular points (which are characterized by non-smoothness of the function Φ) the strain vector can be specified as an arbitrary positive linear combination of normal vectors to the adjacent arcs at this point. Thus, the relations of type given by eqns (7) remain valid at non-regular points if the product is interpreted as the scalar product of appropriate vectors.

Various support conditions should be considered in the further analysis. However, as it was mentioned in Ref. [7], strict boundary values of state variables should not be available when establishing necessary optimality conditions.

From a mathematical point of view, it is not clear, neither does the optimal solution of the problem described by eqns (1)–(3) and (4)–(7) exist, nor is it unique. It is not straightforward to establish the requirements of existence and uniqueness. Resorting to the results of convex analysis, an attempt of this kind was made in Ref. [27]. In this paper we assume that the optimization problem has a physical interpretation from which yields existence of the optimal solution. Therefore, no attention will be paid to this side of the task.

In this study the admissible class of solutions is restricted to the continuous and piece-wise continuous functions. Although the shells with stiffeners are also studied, the number of stiffeners is assumed to be specified. Thus, the problems which lead to the design with an infinite number of infinitely thin stiffeners[28] are outside the scope of this paper.

4. NECESSARY CONDITIONS

Necessary conditions for optimality will be derived by means of variational methods of the optimal control theory. The variables

$$y_1 = M; \qquad y_2 = \frac{dM}{dx}; \qquad y_3 = W; \qquad y_4 = \frac{dW}{dx}; \quad y_5 = U \tag{9}$$

will be referred to as state variables which, according to eqns (4), (5) and (7) must satisfy the following state equations

$$\frac{dy_1}{dx} = y_2; \quad \frac{dy_2}{dx} = +\lambda^2 N\Phi_1 - \frac{N_0}{A} + P;$$
$$\frac{dy_3}{dx} = y_4; \quad \frac{dy_4}{dx} = \lambda^2\Phi_1; \quad \frac{dy_5}{dx} = -\frac{1}{2}y_4^2 + \lambda^2\Phi_2. \tag{10}$$

The yield force N_0 in eqns (10) will be considered as a function depending on H only. For instance, in the case of homogeneous shells $N_0 = \sigma_0 H$, σ_0 being the specified yield stress.

Note that the set of eqns (10) comprises merely the differential constraints. The functional constraints, eqns (2) and (6), imposed on the state variables may be written as

$$R(P,H,y_3)+\theta_1^2 = 0, \qquad \Phi(y_1,N,H)+\theta_2^2 = 0 \tag{11}$$

where the quantities θ_1 and θ_2 are slack variables.

Finally, the phase (state) coordinates must meet the local restrictions (3) and the appropriate boundary conditions. Constraints (3) may be converted into the form

$$g_j + r_j^2 = 0, \qquad j = 1,\ldots,m \tag{12}$$

where r_j are to be regarded as the previously unknown constant parameters (constant slack variables). The boundary and "intermediate" conditions associated with the support conditions for the special problem at hand may be presented as

$$y_i(0) = y_{0i}, \qquad y_j(s_k) = y_{skj}, \qquad y_m(l) = y_{lm}. \tag{13}$$

It is assumed that $i \in I_1$, $j \in I_{2k}$, $m \in I_3$ here. The sets I_1, I_{2k}, I_3 are certain subsets of the set $(1,2,3,4,5)$. If, for instance, the left end of the shell is simply supported, then $y_1(0) = y_3(0) = 0$. It means that $I_1 = (1,3)$ in this case. Alternatively, in case of the completely free right-hand end one has $I_3 = (1,2)$ as now $y_1(l) = y_2(l) = 0$.

It is worth noting that the state variables y_1–y_5 are assumed to be continuous at each point $x \in (0,1)$ except at $x = s_k$ $(k = 1,\ldots,n)$ where the moment and deflection slopes may have finite discontinuities. The finite jumps of y_2 and y_4 occur due to the plastic hinge circles which may crop up at the cross-sections where the additional supports (rigid stiffeners) are located.

In order to establish necessary optimality conditions for the problem (1), (10)–(13) consider an augmented functional[29, 30]

$$\begin{aligned} I_* = G &+ \int_0^l \left(\sum_{i=1}^{5} \psi_i \frac{dy_i}{dx} - L\right) dx + \sum_{i\in I_1} \mu_i(y_i(0)-y_{0i}) \\ &+ \sum_{k=1}^{n}\sum_{i\in I_{ik}} \nu_{ik}(y_i(s_k)-y_{ski}) + \sum_{i\in I_3} \rho_i(y_i(l)-y_{li}) \\ &+ \sum_{j=1}^{2q} \varphi_{1j}(y_2(\alpha_j)-y_{2j}) + \sum_{j=1}^{2r} \varphi_{2j}(y_4(\beta_j)-y_{4j}) \\ &+ \sum_{j=1}^{m} \pi_j(g_j+r_j^2). \end{aligned} \tag{14}$$

Here ψ_i, μ_i, ν_{ik}, ρ_i, φ_{1j}, φ_{2j}, π_j are the Lagrangian multipliers (ψ_1–ψ_5 being the adjoint variables), whereas L is the Hamiltonian (Lagrangian) function defined as

$$\begin{aligned} L = -F + \psi_1 y_2 + \psi_2 + (\lambda^2 N\Phi_1 + (-1)N_0/A + P) + \psi_3 y_4 \\ + \psi_4\lambda^2\Phi_1 + \psi_5(\lambda^2\Phi_2 - y_4^2/2) + \varphi(R+\theta_1^2) + \chi(\Phi+\theta_2^2). \end{aligned} \tag{15}$$

The sums including the coefficients φ_{1j} and φ_{2j} in eqn (14) are associated with state constraints (2) and (6) (or (11) in another form), where y_{2j} and y_{4j} are assumed to be the given constants. If quantity H must not be regarded as a control function, these restrictions have to be conceived as the second-order constraints imposed on the state variables[29, 30]. The contrary version will be discussed later.

Various approaches to the optimal control problems with state variable inequality constraints are presented in Refs [29–36]. As a rule, the control technique depends on the order of the inequality constraint[29, 36]. An optimization procedure for optimal control problems with the second-order inequality constraints is developed in Refs [7, 11, 12]. It will be applied in this paper as well.

Let us assume that

$$\Phi = 0 \quad \text{for} \quad x \in (\alpha_{2i-1}, \alpha_{2i})$$

and

$$R = 0 \quad \text{for} \quad x \in (\beta_{2j-1}, \beta_{2j})$$

where $i = 1, \ldots, q$ and $j = 1, \ldots, r$.

Differentiating these relations with respect to x one eventually obtains the intermediate values of variables y_2 and y_4

$$\begin{aligned} y_{2i} &= -\frac{\partial \Phi}{\partial H}\frac{\partial H}{\partial x}\left(\frac{\partial \Phi}{\partial y_1}\right)^{-1}\Bigg|_{x \times \alpha_i}, \qquad i = 1, \ldots, 2q; \\ y_{4j} &= -\left(\frac{\partial R}{\partial P}\frac{\partial P}{\partial x} - \frac{\partial R}{\partial H}\frac{\partial H}{\partial x}\right)\left(\frac{\partial R}{\partial y_3}\right)^{-1}\Bigg|_{x=\beta_j}, \qquad j = 1, \ldots, 2r \end{aligned} \tag{16}$$

which appeared in eqn (14).

For the optimality of the solution it is necessary that the total variation ΔI_* of functional (14) should equal zero. Required variations should be determined by the following sample

$$\begin{aligned} \Delta \int_0^l y \, \mathrm{d}x &= \int_0^l \delta y \, \mathrm{d}x - \sum_{i=1}^{n} [y(s_i)] \Delta s_i \\ \Delta y(s \pm 0) &= \delta y(s \pm 0) + \frac{\mathrm{d}y(s \pm 0)}{\mathrm{d}x} \Delta s \end{aligned} \tag{17}$$

where δy is the weak variation of y and Δs stands for an increment of the constant parameter s. Brackets denote the finite discontinuity of the corresponding variable, e.g.

$$[y(s)] = y(s+0) - y(s-0); \qquad y(s \pm 0) = \lim_{x \to s \pm 0} y(x). \tag{18}$$

Variation of functional (14) which has to be performed making use of eqns (15)–(18) yields the conditions of optimality. These include two respective groups of relations for the determination control functions and the parameters accompanied by the adjoint equations with the transversality conditions.

The conditions for control functions and parameters are found to be respectively

$$\varphi \theta_1 = 0, \qquad \chi \theta_2 = 0, \qquad \lambda(\Phi_1(\psi_4 + N\psi_2) + \psi_5 \Phi_2) = 0 \tag{19}$$

and

$$\pi_j r_j = 0, \qquad j = 1, \ldots, m$$

$$\frac{\partial G_*}{\partial s_k} + [L(s_k) + F(s_k)] = 0, \qquad k = 1, \ldots, n \tag{20}$$

$$\frac{\partial G_*}{\partial p} - \int_0^l \frac{\partial L}{\partial P}\frac{\partial P}{\partial p}\,\mathrm{dx} = 0, \qquad \frac{\partial (G_* - y)}{\partial h} - \int_0^l \frac{\partial L}{\partial H}\frac{\partial H}{\partial h}\,\mathrm{d}x = 0$$

where the following notations are used:

$$Y = \sum_{k=1}^{n} \sum_{i \in I_{2k}} v_{ik} y_{ski} + \sum_{i \in I_1} \mu_i y_{0i} + \sum_{i \in I_3} \rho_i y_{li} + \sum_{j=1}^{2q} \varphi_{1j} y_{2j} + \sum_{j=1}^{2r} \varphi_{2j} y_{4j}, \tag{21}$$

$$G_* = G + \sum_{j=1}^{m} \pi_j g_j.$$

The adjoint system has the traditional form

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}x} = -\frac{\partial L}{\partial y_i}, \qquad i = 1, \ldots, 5. \tag{22}$$

Finally, the transversality conditions are at the point $x = 0$

$$\begin{aligned} \psi_i(0) &= 0, \qquad i \bar{\in} I_1, \\ \psi_i(0) &= \mu_i, \qquad i \in I_1 \end{aligned} \tag{23}$$

at the intermediate points $x = s_k$ $(k = 1, \ldots, n)$

$$\begin{aligned} \psi_i(s_k) &= 0, \qquad i \bar{\in} I_{2k}, \qquad [y_i(s_k)] \neq 0, \\ [\psi_i(s_k)] &= 0, \qquad i \bar{\in} I_{2k}, \qquad [y_i(s_k)] = 0, \\ [\psi_i(s_k)] &= v_{ik}, \qquad i \in I_{2k} \end{aligned} \tag{24}$$

at the end point $x = 1$

$$\begin{aligned} \psi_i(l) &= 0, \qquad i \bar{\in} I_3, \\ \psi_i(l) &= -\rho_i, \qquad i \in I_3 \end{aligned} \tag{25}$$

and at the intermediate points $x = \alpha_j$, $x = \beta_j$, $x = x_j$

$$\begin{aligned} &[\psi_2(\alpha_j)] = \varphi_{1j}, \qquad [\psi_i(\alpha_j)] = 0, \qquad i \neq 2, \qquad j = 1, \ldots, 2q, \\ &[\psi_4(\beta_j)] = \varphi_{2j}, \qquad [\psi_i(\beta_j)] = 0, \qquad i \neq 4, \qquad j = 1, \ldots, 2r, \\ &[\psi_i(x_j)] = \sum_{k=1}^{m} \pi_k \frac{\partial g_k}{\partial y_i}. \end{aligned} \tag{26}$$

It is worth emphasizing that eqns (19)–(26) are derived under the assumption that $P = P(x, p)$ and $H = H(x, h)$ are the specified functions, p and h being the unknown constant parameters. If these functions are not given and it is known *a priori* that they must not be handled as constants, an additional analysis is necessary.

Assume now alternatively that H is a completely unspecified function. Henceforth it must be referred to as a control variable. Hence the order of state constraints (2) and (6)

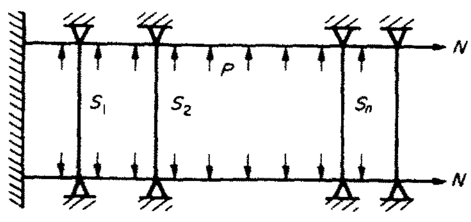

Fig. 3. Shell with rigid stiffeners.

is equal to zero (not two as previously). This in turn, demands that eqn (14) must be slightly changed because now the series with multipliers φ_{1j} and φ_{2j} is unnecessary. Setting these to zero in eqns (26), one obtains

$$[\psi_i(\alpha_j)] = [\psi_i(\beta_k)] = 0; \quad i = 1,\ldots,5; \quad j = 1,\ldots,2q; \quad k = 1,\ldots,2r \tag{27}$$

which yields the continuity of adjoint variables, if no additional support is applied. Moreover, since H is a control variable the last equation in (20) must be substituted by

$$\frac{\partial L}{\partial H} = 0. \tag{28}$$

Equation (28) represents the principle of maximum in the case of the unbounded set of admissible values of the control function. A problem of this type related to a close shell is studied in Ref. [9].

The optimality conditions similar to eqns (19)–(26) are discussed in greater detail in Refs [7, 13]. In particular, it was pointed out that the optimal trajectory in the state space comprises the ordinary as well as singular subarcs[37].

5. APPLICATIONS

5.1. *Optimal location of additional supports*

As the first illustration of the previous analysis can serve as a problem consisting in the determination of the optimal positions of rigid stiffeners (additional supports). Consider a closed cylindrical shell clamped at the left-hand end and simply supported at the right end (Fig. 3). The closed shell should be conceived as a structure consisting of a cylindrical shell and of two end plates which are fixed as described above. At the preliminary unspecified positions $x = s_1, \ldots, s_n$ additional supports are located. Let the shell be subjected to the internal pressure P, which is constant with respect to the coordinate x.

Such a layout of the rigid circular supports (which would be regarded as rigid stiffeners) is sought for which minimizes the optimality criterion

$$I = \int_0^l y_3 \, dx. \tag{29}$$

Since the post-yield response of the shell is expected to take place, the load intensity has to meet requirements

$$P - P_1 \geqslant 0, \ldots, P - P_{n+1} \geqslant 0. \tag{30}$$

In (30) P_j stands for the load carrying capacity of the part of the shell, which is located between supports s_{j-1} and s_j $(j = 1,\ldots,n+1)$, where $s_0 = 0$, $s_{n+1} = 1$.

The problem described herein could be considered as a particular case of eqns (1)–(3). In fact, it is associated with $G = 0$, $F = W$, $R = 0$, $g_j = -P + P_j$, $m = n+1$, where P and H are the given constants.

The boundary conditions for state variables are the following:

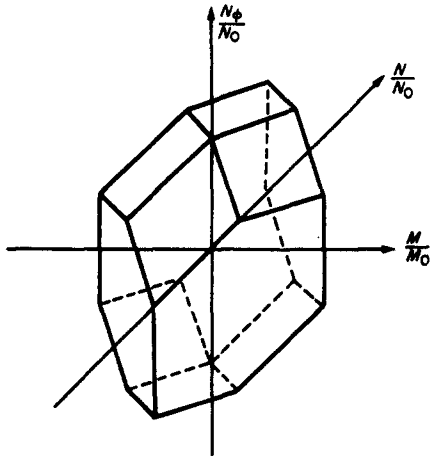

Fig. 4. Yield surface for a sandwich Tresca shell.

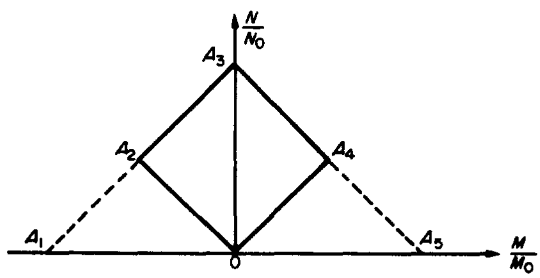

Fig. 5. Sketch of the face $N_\varphi = N_0$ of the yield polyhedron.

$$\begin{aligned} &y_1(s_j) = M_*, \qquad y_1(l) = 0, \\ &y_3(s_j) = y_3(l) = 0 \qquad j = 0, \ldots, n \\ &y_5(0) = 0 \end{aligned} \tag{31}$$

where M_* is the value of the bending moment on the plastic hinge circles. It is assumed that the hinges occur at the cross-sections where the additional supports are located. Evidently M_* depends on the form of the yield condition which has to be employed.

This problem was examined in detail in Ref. [12] using the concept of a sandwich shell. The yield surface corresponding to the sandwich cylindrical shell made of a material obeying the Tresca yield condition is presented in Fig. 4. Since we restricted our attention to the face $N_\varphi = N_0$ of the yield polyhedron (Fig. 4), the yield curve would be presented as diamond $0A_2A_3A_4$ in Fig. 5. It could be shown, that in the case of a closed shell

$$\frac{N}{N_0} = \frac{AP}{2N_0} > \frac{1}{2}.$$

Thus, function Φ may be specified as

$$\Phi = \left|\frac{M}{M_0}\right| + \frac{N}{N_0} - 1 \leqslant 0 \tag{32}$$

where $M_0 = \sigma_0 Hh$, $N_0 = 2\sigma_0 h$, h and H being the constant face sheet thickness and the thickness of the shell, respectively.

Taking eqns (29)–(32) into account the optimality conditions (20) may be converted into

Table 1. Optimal positions of the rigid stiffener

p	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00
s_1	0.520	0.513	0.510	0.509	0.505	0.503	0.502	0.500
e	0.6201	0.9896	0.9963	0.9985	0.9994	0.9998	0.9999	1.0000

$$\begin{aligned} &\pi_j r_j = 0, && j = 1,\ldots,n+1; \\ &[L(s_i)] = -\sum_{j=1}^{n+1} \pi_j \frac{\partial P_j}{\partial s_i}, && i = 1,\ldots,n \end{aligned} \tag{33}$$

which coincide with those obtained in Ref. [12]. Requirements (22)–(26) related to the adjoint variables hold good in this particular case if $\varphi_{2j} = 0$ ($j = 1,\ldots,2r$). The sets I_1, I_{2k}, I_3 may be adopted according to conditions (31) and (32).

Equations (33) serve for determination of the optimal locations of additional supports. Naturally, before that one has to specify the adjoint coordinates according to eqns (22)–(26) and integrate the state equations (10) making use of eqns (31) and (32). Finally one obtains[12]

$$\frac{s_j}{l} = \begin{cases} j\sqrt{\left(\dfrac{8(2-p)}{\omega(p-1)}\right)}, & p_0 \leqslant p \leqslant p_1 \\ \dfrac{j}{n^2-1}\left\{n - \sqrt{\left(1 - \dfrac{(n^2-1)(2-p)}{\omega(p-1)}\right)}\right\}, & p_1 \leqslant p \leqslant 2 \end{cases} \tag{34}$$

where $j = 1,\ldots,n$ and

$$\begin{aligned} &p = \frac{AP}{N_0}, \quad \omega = \frac{N_0 l^2}{AM_0}, \quad p_{0,1} = 1 + \frac{1}{1+\omega a_{0,1}}, \\ &a_0 = (1+(1+2n)\sqrt{2})^{-2}, \quad a_1 = \frac{1}{2}\left(\frac{4n-\sqrt{14}}{8n^2-7}\right)^2. \end{aligned} \tag{35}$$

It follows from eqns (34) that in the cases associated with $p = p_0$ and 2

$$s_j(p_0) = \frac{2\sqrt{2j}}{1+(1+2n)\sqrt{2}}, \quad s_j(2) = \frac{j}{n+1}, \quad j = 1,\ldots,n. \tag{36}$$

Consequently, for $p = p_0$ the layout of the additional supports corresponds to the maximum load carrying capacity, p_0 being the limit load for each part of the shell. Note that this result was also observed in the cases of shells and beams subjected to impulsive loading[38, 39].

The upper bound of the pressure ($p = 2$) in eqns (34) is associated with the onset of the membrane stage of loading. According to eqns (36), the additional supports must now be located so that the distances between them are equal.

Economy of the design established could be assessed by the ratio

$$e = \frac{I(s_0)}{I(s_u)} \tag{37}$$

where $I(s_0)$ stands for the optimal value of functional (29) and $I(s_u)$ is the value of functional (29) associated with the uniform layout of stiffeners. In the case $n = 1$, the values of the economy ratio (37) are given in Table 1, which correspond to $\omega = 16$ and $p_0 = 1.63$.

Calculations carried out reveal that the economy coefficient e approaches zero when the intensity of the pressure loading tends to the load carrying capacity from above. This,

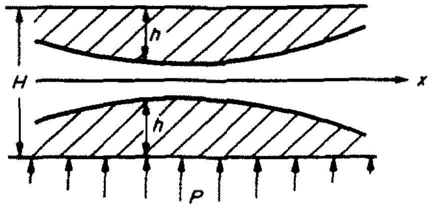

Fig. 6. Longitudinal shell wall section.

however, does not imply that a considerable amount of economy may be achieved by using a finite number of stiffeners. It was established in Ref. [40] that about 23% of the material should be saved by the utilization of stiffeners in the geometrically linear case. On the other hand, in Ref. [40] as well as in Ref. [41] the volume of stiffeners is taken into account. The material consumption of stiffeners is disregarded (these are conceived as rigid ring supports) but the effect of geometrical nonlinearity, e.g. changes of configuration, is included in this paper. In the limit state of the shell associated with the load carrying capacity we have $W = y_3 = 0$. Thus, for the optimal set of stiffeners $I(s_0(p_0)) = 0$, but $I(s_u(p_0)) \neq 0$ (because now $y_3 \neq 0$), from where yields $e = 0$.

5.2. *Optimal design for a specified deflected shape*

Consider now an open cylindrical shell hinged at both ends which is subjected to the uniformly distributed internal pressure P and to the specified axial tension N. The optimal thickness variation is sought for under the requirement of minimum material consumption and the deflection of the optimal shell should not be greater than that of the constant thickness shell. The shell is assumed to be of sandwich cross-section with variable face-sheet thickness (Fig. 6). No stiffener is utilized.

One has to minimize the functional

$$I = \int_0^l h \, dx \tag{38}$$

provided $W - W_* \leqslant 0$, where W_* is the deflection of the shell of constant thickness h_*. Thus, the special case of eqns (1)–(3) associated with $G = 0$, $F = h$, $g_j = 0$ will be examined.

It is worth emphasizing that h is a variable quantity whereas H is a given constant now. It is expected that the alteration of the contributions of h and H does not produce any confusion.

For the sake of simplicity we confine ourselves to the approximate yield surface which circumscribes on the exact Tresca surface and corresponds to the configuration $0A_1A_2A_3A_4A_5$ in Fig. 5. Thus, relation (32) holds good for each value of $N \leqslant N_0$ in the present case.

The problem set up above was investigated in Refs [9, 10]. It is straightforward to check whether the optimality conditions (and the final results) established in Ref. [10] follow from relations (19)–(28) if relations (15), (32) and (38) are taken into account. In fact, since the boundary conditions are

$$y_1(0) = y_1(l) = 0, \quad y_3(0) = y_3(l) = 0, \quad y_2(\tfrac{1}{2}) = 0, \quad y_4(\tfrac{1}{2}) = 0, \quad y_5(0) = 0 \tag{39}$$

one readily obtains from relations (22)–(27) that $\psi_5 = 0$. Therefore, eqns (19) may be represented as

$$\theta_1 = 0, \quad \theta_2 = 0, \quad \lambda(\psi_3 + N\psi_1) = 0, \quad \lambda(\psi_4 + N\psi_2) = 0 \tag{40}$$

whereas eqns (20) must be substituted by eqn (28).

Table 2. Optimal face-sheet thickness

ξ	n_* 0.00	0.20	0.40	0.60
0.0	1.101	1.049	1.020	1.0057
0.1	1.093	1.050	1.021	1.0059
0.2	1.069	1.053	1.022	1.0062
0.3	1.027	1.046	1.024	1.0068
0.4	0.967	1.015	1.027	1.0077
0.5	0.885	0.960	1.014	1.0089
0.6	0.778	0.878	0.969	1.0104
0.7	0.642	0.767	0.889	0.9873
0.8	0.472	0.621	0.770	0.9135
0.9	0.261	0.434	0.610	0.7861
1.0	0.000	0.200	0.400	0.6000

Table 3. Volume rations

	n_* 0.0	0.2	0.4	0.6	0.8
$\omega^2 = 4, p_*$	1.50	1.40	1.30	1.20	1.10
e_*	0.7770	0.8216	0.8662	0.9108	0.9554
$\omega^2 = 8, p_*$	1.25	1.20	1.15	1.10	1.05
e_*	0.8111	0.8489	0.8867	0.9244	0.9622

Requirements (40) imply that (i) the equality sign can be applied in relation (32) at each point of the shell, (ii) the deflections of the optimal design and of the constant thickness shell coincide. These conclusions are consistent with the results established in Refs [9, 10].

The optimal face-sheet thickness is found to be

$$v = \begin{cases} 1+\{n_*-p^*+(p^*-1)\,\mathrm{ch}\,(\omega(1-\xi_1))\}\dfrac{\mathrm{ch}\,\omega\xi}{\mathrm{ch}\,\omega}, & (0,\xi_1) \\ p_*+\dfrac{1}{\mathrm{ch}\,\omega}\{(n_*-p_*)\,\mathrm{ch}\,\omega\xi+(1-p_*)\,\mathrm{sh}\,\omega\xi_1\,\mathrm{sh}\,(\omega(1-\xi))\}, & (\xi_1,1). \end{cases} \tag{41}$$

Due to symmetry, the thickness distribution is represented for the right-hand side of the shell only. In eqns (41) the following notations are used:

$$\xi = \frac{2x}{l}-1,\quad n_* = \frac{N}{N_*},\quad p_* = \frac{PA}{N_*},\quad \omega = \sqrt{\left(\frac{N_* l^2}{4AM_*}\right)},\quad v = \frac{h}{h_*}.$$

Here N_* and M_* denote the yield force and yield moment for the constant thickness shell, respectively, and

$$\xi_1 = 1-\sqrt{\left(\frac{2(1-n_*)}{\omega^2(p_*-1)}\right)}. \tag{42}$$

Values of the non-dimensional thickness v are given in Table 2. It is worth mentioning that in the central part of the shell the thickness differs slightly from that corresponding to the reference shell of constant thickness. The exposed values of the volume correspond to the case, where $\omega = 2$ and $\xi_1 = n_*$.

Economy of the design of eqns (41) and (42) should be assessed by the coefficient $e_* = I(v)/(h_* l)$. The specific values of this ratio associated with the limit state ($\xi_1 = 0$) are accommodated in Table 3. The first two lines in Table 3 correspond to the shell with $\omega = 2$, the last two to $\omega = 2\sqrt{2}$. The results show that a considerable economy is available for small values of the axial tension.

6. CONCLUDING REMARKS

An optimal design method developed for rigid–plastic cylindrical shells is outlined in this paper. The essential point of the study is that the variational methods of the optimal control theory are employed in deriving optimal projects in the case of plastic shells operating beyond the incipient yield-point load. Attention is focused on an ideal rigid–plastic material. It is assumed that the stress–strain state of the shell corresponds to the face $N_\varphi = N_0$ of the yield surface in the space of stress resultants. The latter assumption introduced for the sake of simplicity is not a substantial requirement. As it was shown in Ref. [13], the development of a similar parametrical optimization technique is possible by relaxed assumptions.

Two particular problems are discussed in greater detail. The first example consists in the determination of optimal positions for rigid stiffeners, whereas the other example presents a minimum volume problem for the specified deflected shape. The method used in this paper seems to be applicable for quite a broad scale of problems. Evidently, it should be readily modified for minimum volume problems, for instance, which refer to the cylindrical shells of piece-wise constant thickness as well as to the rib-reinforced tubes.

The optimization procedure described herein results in a unified approach to the optimal design of thin-walled rigid–plastic cylindrical shells including the effect of geometry changes. It is useful, first of all, in the case of a piece-wise linear approximation of the exact yield surface. The use of a non-linear yield surface involves computer programs for solving non-linear boundary value problems. This is, however, the subject of a subsequent work.

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